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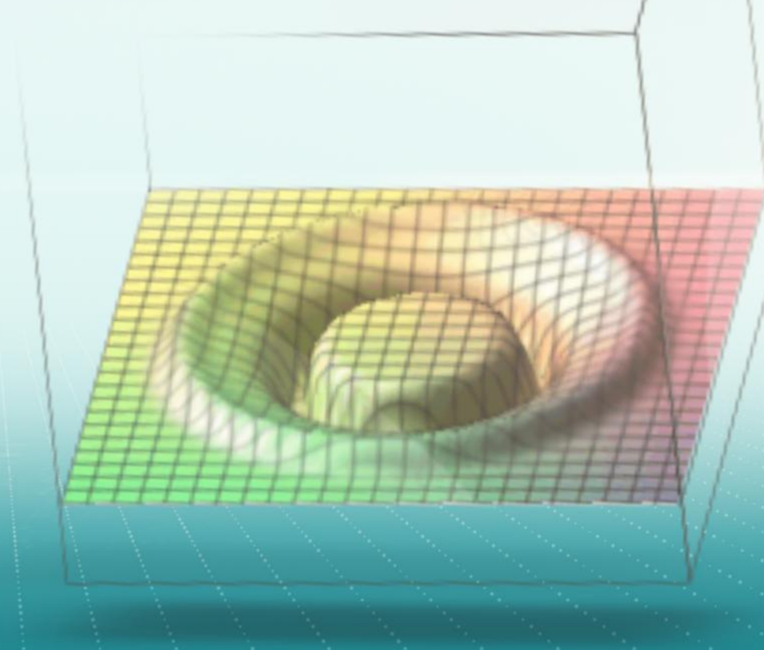

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ABSTRACT

1

For the first time the exact vortex solution of the Cauchy problem in unbounded space is obtained for the three-dimensional Euler-Helmholtz (EH) equation in the case of a nonzero-divergence velocity field for an ideal compressible medium.

The solution obtained describes the inertial vortex motion and coincides with the exact solution to the three-dimensional Riemann-Hopf (RH) equation which simulates turbulence without pressure [S. Chefranov, 1991].

2

A necessary and sufficient condition of the onset of a singularity in the evolution of the enstrophy in finite time for EH equation is obtained.

3

A closed description of the evolution of the enstrophy and the all other moments of the velocity and vortex fields is given, i.e., the main problem of theory of turbulence is solved exactly on the base of analytic solution of EH equation.

4

A new analytic solution of the Cauchy problem for the three-dimensional Navier-Stokes equation is obtained. This solution coincides with the above-mentioned smooth solution to the EH and RH equations, which take into account the viscosity effect of a compressible medium and also the sufficient condition of positive definiteness of the growth rate of the entropy in the form of a linear relation between the pressure and the divergence of the velocity field. This gives the positive solution to one of the Millennium Prize Problems even for the solution of the compressible Navier-Stokes equation (www.claymath.org).

The Euler equations, which express the impulse and mass conservation laws, for the case of ideal compressible medium are well-known for already more than 250 years (since 1755) and have the following form:

1.1 $$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = \frac{1}{\rho}\left(f_i - \frac{\partial p}{\partial x_i}\right)$$

1.2 $$\frac{\partial p}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0$$

where according to the repeated indices meant is a summation from 1 to n (n is a dimension of space), and $u_i; f_i; p; \rho$ the components of the velocity field, the external forces field, field of pressure and density, respectively...

Equation (1.1) is also the Navier-Stokes equation for viscous compressible medium, if for the force f_i in (1.1) the following representation takes place:

1.3 $$f_i = \eta \frac{\partial^2 u_i}{\partial x_k^2} + (\varsigma + \frac{\eta}{3}) \frac{\partial}{\partial x_i} (\frac{\partial u_k}{\partial x_k})$$

In (1.3) η; ς are the constant coefficients of viscosity and the second viscosity, respectively.

Since 2000 the problem of the existence of smooth solutions for the three-dimensional (3D) Navier-Stokes equation is one of the seven fundamental problems for the Millennium Prize formulated at the Clay Mathematical Institute (CMI). In CMI his problem is formulated not for the general form of the Navier-Stokes equation (1.1), (1.3), but for the special case when the approximation of an incompressible medium $\rho = const$ is assumed to be fulfilled in (1.1) - (1.3) for zero divergence of the velocity field

1.4 $$div\vec{u} \equiv \frac{\partial u_k}{\partial x_k} = 0$$

A necessary condition for carrying out this approximation is the assumption that the Mach numbers $Ma = \frac{|\vec{u}|}{c} \ll 1$ are small (where c is the speed of sound in a given medium).

Such a formulation is connected not only with the explicit simplification of the form of system (1.1) - (1.3). The main thing here is the a priori idea that the complete system (1.1) - (1.3) cannot have smooth solutions on an arbitrarily large time interval. The reason for this is the possibility of appearance of a singularity (collapse) arising in the solution in a finite time $t_0 > 0$, as, for example, in case of collapse of a traveling nonlinear wave in an ideal compressible medium.

Euler also noted the complexity of the analysis necessary to obtain the general form of the solution of system (1.1), (1.2) and pointed out the importance of obtaining at least particular solutions of these equations. Thus, for example, in Euler 1755 considered is the solution corresponding to an exact hydrostatic equilibrium, when in (1.1) the total force on the RHS of equation (1.1) is equal to zero.

Following this logic of Euler's work, and also under the condition of zero balance of forces on the RHS of the Euler equation (1.1), we can consider that despite the velocity is constant for each particles, it is not necessarily to be the same for different particles of the medium:

1.5 $$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} = 0, i = 1, \dots, n$$

This equatioт describes inertial motion of the medium particles and is the n- dimensional generalization of the well-known one-dimensional equation for nonlinear travelling wave, sometimes called as the Hopf equation.

The closed system of equations of type (1.5), (1.2) is also obtained from the equations of hydrodynamics of self-gravitating dust matter when considering the formation of a large-scale structure of the Universe under the assumption of zero pressure $p = 0$ in (1.1) Ya.Zel′dovich 1970.

After applying the curl operator to the Hopf equation (1.5) the Euler vortex 3D equation (also called the Helmholtz vortex equation) follows from it and has the form:

1.6 $$\frac{\partial \omega_i}{\partial t} + u_k \frac{\partial \omega_i}{\partial x_k} = \omega_k \frac{\partial u_i}{\partial x_k} - \omega_i \frac{\partial u_k}{\partial x_k}$$

where $\vec{\omega} = rot\vec{u}$. Exactly the same equation is obtained when applying the curl operation to initial Euler equation (1.1) (if in (1.1) $rot\vec{f} = 0$ and $rot\left(\frac{1}{\rho}\vec{\nabla}p\right) = 0$

Thus, the nonstationary vortex solution of equation (1.5) definitely determines the form of the equation (1.6) solution for the vortex field of the compressible ideal medium flow.

However, all well-known solutions of the Hopf 3D equation (1.5) are obtained only in Lagrange variables (Zel'dovich, 1970; Kusnetzov, 2003). Therefore they do not give the solution for the Euler vortex 3D equation (1.6) for the vortex field just in Euler variables.

The general exact vortex solution of equations (1.5), (1.6) was obtained in Euler variables only in 1991 (Chefranov, 1991).

On the basis of this solution an analytic solution of the Navier-Stokes equation (1.1), (1.3) for a compressible medium is obtained with a model accounting of the friction force effect. This is necessary for the regularization of solution (1.5) and (1.6) that loses its smoothness in a finite time t_0 of the solution collapse realization.

In particular, it is shown that when choosing the coefficient of uniform friction (when introducing member $-\mu u_i$ into the RHS of the Hopf equation (1.5), which satisfies the criterion:

1.7 $$\mu > \frac{1}{t_0}$$

the solution already becomes regular and keeps smoothness for arbitrarily large intervals of time of the solution evolution. In condition (1.7) the value of the friction coefficient is equal to $\mu = \frac{\nu}{\iota_{min}^2}$, where ν is the coefficient of kinematic viscosity of the medium, and $\iota_{min} \propto 1/k_{max}$ is the characteristic scale, associated, for example, with the inevitable cutoff wavenumbers at some maximum number k_{max}, that is typical for any numerical simulations on the basis of equation (1.1).

Thus, only when we take into account the regularity condition (1.7), it is possible to avoid the occurrence of instabilities caused by the loss of smoothness of solutions in finite time, that allows us to qualitatively and quantitatively expand the predictability limits on the basis of the corresponding numerical calculations using the hydrodynamic equations. Up to now, numerical calculations based on the Euler equation did not take into account the need to establish a correlation between the initial data used, which determine the value of t_0 in (1.7), and the value of the cutoff scale which determines the value of μ in the left side of inequality (1.7).

It is also shown that when modeling viscous forces by introducing a random Gaussian field $\vec{V}(t)$ (with the substitution of $u_i(\vec{x},t) \to u_i(\vec{x},t) + V_i(t)$ in (1.5)), the regularization of the singular modes in the evolution of the solutions of the vortex 3D Euler equation (1.6) is already achieved for any arbitrarily small effective viscosity coefficient ν, determined in the form

1.8 $$\langle V_i(t+\tau)V_j(t)\rangle = 2\nu\delta_{ij}\delta(\tau)$$

where the angular brackets denotes the statistical averaging, δ is a Dirac delta-function, and δ_{ij} is a unity matrix.

ENERGY AND ENTROPY BALANCE EQUATION

Integral kinetic energy balance equation for the compressible medium

2.1 $$E = \frac{1}{2}\int d^n x \rho u^2 \frac{dE}{dt} = -\eta \int d^n x (\frac{\partial u_i}{\partial x_k})^2 + \int d^n x \left[p - \left(\zeta + \frac{\eta}{3}\right) div\vec{u}\right] div\vec{u}$$

This expression under condition (1.4) exactly coincides with the given in [3] (see (16.3) in [3]) in case of incompressible medium and serves as its generalization for the compressible medium flow.

From the condition of conservation of total energy [3] $E_n = \int d^3 x (\rho \frac{\vec{u}^2}{2} + \rho\varepsilon)$ and the balance equation (ε is a density of the internal energy; see derivations in [16]):

2.2 $$\frac{\partial}{\partial t}\left(\rho\frac{\vec{u}^2}{2} + \rho\varepsilon\right) = -\frac{\partial}{\partial x_k}\left[u_k\left(\rho\left(\frac{\vec{u}^2}{2} + \Phi_0\right) + \rho - \left(\zeta + \frac{\eta}{3}\right) div\vec{u}\right) - \eta\frac{\partial}{\partial x_k}\left(\frac{\vec{u}^2}{2}\right)\right] + T\left(\frac{\partial}{\partial t}(\rho s) - \frac{B}{T}\right),$$

$$B = \eta(\frac{\partial u_i}{\partial x_k})^2 - \left[p - \left(\zeta + \frac{\eta}{3}\right) div\vec{u}\right] div\vec{u}$$

follows the balance equation for density (per unit mass) of entropy s

2.3 $$\frac{\partial}{\partial t}(\rho s) = \frac{B}{T}$$

where T is the temperature.

From (2.3) we obtain the following equation of the integral entropy $S = \int d^3 x\rho s$ balance:

2.4 $$\frac{d}{dt}S = \eta \int d^3 x \frac{1}{T} (\frac{\partial u_i}{\partial x_k})^2 - \int d^3 x \frac{1}{T} div\vec{u} \left[p - \left(\zeta + \frac{\eta}{3}\right) div\vec{u}\right]$$

Comparing the form of equations (2.4) and (2.1), for the considered isothermal case (when $T = T_0 = const$), we obtain the following exact relation (Landau, Lifshitz, 1986):

2.5 $$T_0 \frac{dS}{dt} = \frac{dE}{dt}$$

This additional equation is obtained from a sufficient condition for the positive definiteness of the integral entropy (2.4) growth rate, which has the following form:

2.6 $$p = \left(\varsigma + \frac{\eta}{3}\right) div\vec{u}$$

Equation (2.6) is an additional equation to system (1.1)-(1.3) and makes it closed.

1

Let us find the solution of closed system (1.1) - (1.3) and (2.6) on the basis of the exact solution of the Hopf equation (1.5) satisfying the vortex form of the Euler equation (1.6) obtained in [15] and having the following form (see also [16]):

3.1 $$u_i(\vec{x},t) = \int d^n\xi u_{0i}(\vec{\xi})\delta(\vec{\xi} - \vec{x} + t\vec{u}_0(\vec{\xi}))\det\hat{A}$$

where $\det\hat{A}$ is a determinant of matrix $\hat{A} \equiv A_{nm} = \delta_{nm} + t\frac{\partial u_{0n}}{\partial \xi_m}$, and $u_{0i}(\vec{x})$ is an arbitrary initial smooth velocity field. This solution preserves smoothness only for time $t < t_0$, for which condition $\det\hat{A} > 0$ is executed, where t_0 is a minimal time, in space coordinates, for which determinant $\det\hat{A} = 0$ vanishes.

In particular, for one-dimensional case n=1 we have $det\hat{A} = 1 + t\frac{du_{01}}{d_{\xi 1}}$ and solution (3.1) exactly coincides with the solution obtained in [18, 19]. At the same time $t_0 = \frac{1}{\max|du_0/dx|} > 0$ and, for example, for initial distribution $u_0(x) = a\exp(-x^2/L^2), a > 0$ we have $t_0 = \frac{L}{a}\sqrt{\frac{e}{2}}$, where $x = x_{max} = L/\sqrt{2}$ and the solution singularity can be realized only for $x > 0$.

When introducing into equation (1.5) the homogeneous friction with coefficient $\mu > 0$ the solution for modified equation (1.5) is obtained from (3.1) if to substitute in (3.1) the new time variable by changing $t \to \tau(t) = (1 - e^{-\mu t})/\mu$. Besides, it is obvious that if condition (1.7) is satisfied, solution (3.1) remains smooth for any arbitrarily large times. Actually, under condition (1.7), even in the limit $t \to \infty$, determinant $\det\hat{A}$ does not vanish, remaining positive, since the condition $\tau(t) < t_0$ is preserved in this limit. At the same time the noted modification of solution (3.1) satisfies vortex 3D equation (1.6) when introducing member $-\mu\omega_i$ into the right side (1.6).

If we replace $u_i \to u_i + V_i(t)$ in (1.5), then solution (3.1) will still satisfy equations (1.5) and (1.6) if we replace $\vec{x} \to \vec{x} - \vec{B}(t), \vec{B}(t) = \int_0^t dt_1 \vec{V}(t_1)$ in (3.1).

If we assume that the velocity field $\vec{V}(t)$ is a random Gaussian field of white noise type satisfying the condition (1.8), then after statistical averaging of this modification of expression (3.1) we obtain for the average velocity an expression that remains smooth for any time intervals and has the following form:

3.2 $$\langle u_i \rangle = \int d^n\xi u_{0i}(\vec{\xi})|\det\hat{A}| \frac{1}{(2\sqrt{\pi\nu t})^2} \exp\left[-\frac{(\vec{x} - \vec{\xi} - t\overrightarrow{u_0}(\vec{\xi}))^2}{4\nu t}\right]$$

Thus, we obtain a smooth for any times solution of the Navier-Stokes equation for a viscous compressible medium in the form (3.2).

For solution (3.1) the vortex field in two-dimensional case exactly satisfies equation (1.6) and has the following form:

3.3
$$\omega(\vec{x},t) = \int d^2\xi\, \omega_0(\vec{\xi})\delta(\vec{\xi} - \vec{x} + t\overrightarrow{u_0}(\vec{\xi}))$$

where $\omega_0(\vec{x})$ is an initial distribution of the vortex field on plane. In case of substitution $\omega_0(\vec{x}) \rightarrow \rho_0(\vec{x})$ expression (3.3) gives the distribution for the density field (not only in two-dimensional, but also in three-dimensional case, if the integration is considered in three-dimensional space in (3.3) and all the vectors appearing in (3.3) are also considered as three-dimensional).

The exact solution of the three-dimensional Euler vortex equation (1.6) corresponding to the velocity field (3.1) has the following form:

3.4
$$\omega_i(\vec{x},t) = \int d^3\xi (\omega_{0i}(\vec{\xi}) + t\omega_{0j}\frac{\partial u_{0i}(\vec{\xi})}{\partial \xi_j})\delta(\vec{\xi} - \vec{x} + t\vec{u}_0(\vec{\xi}))$$

where $\vec{\omega}_0 = rot\vec{u}_0$.

The expression for enstrophy corresponding to exact solution (3.4) has the following form:

3.5 $$\Omega_3 \equiv \int d^3 x \omega_i^2(\vec{x}, t) = \int d^3 \xi (\omega_{0i} + t\omega_{0j} \frac{\partial u_{0i}}{\partial \xi_j})^2 / \det \hat{A}$$

The expressions for any higher moments of the vortex field are obtained similarly. For simplicity's sake, let us give them only for two-dimensional case, when they have the following form:

3.6 $$\Omega_{2(m)} = \int d^2 x \omega^m = \int d^2 \xi \frac{\omega_0^m(\vec{\xi})}{\det^{m-1} \hat{A}};$$

$$\Omega_{2(2m)} = \int d^2 x \omega^{2m} = \int d^2 \xi \frac{\omega_0^{2m}(\vec{\xi})}{\det^{2m-1} \hat{A}};$$

$$m = 1,2,3, \dots$$

Thus, the exact solution of the closure problem, the main problem in the theory of turbulence, is obtained. In particular, from (3.6) it follows that in limit $t \to t_0$ we have inequality $\Omega_{2(m)}^2 \ll \Omega_{2(m)}^2$, which is typical for realization of the high intermittency of turbulence.

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CONCLUSIONS

We establish the possibility of the existence of a smooth solution on an unlimited interval of time for the analytical approximate (only because of the use of the model representation for the viscosity forces) solution of the Navier-Stokes equation describing the flow of a compressible medium. For the pressure field corresponding to this solution, a representation providing a sufficient condition for the positive definiteness of the rate of change in integral entropy is used, which is, as shown below, more correct than the traditional use of the equation of the medium state. As a result, it is shown that a positive answer to the generalization of the Clay Millennium Problem is possible exactly for the case of a compressible medium.

Besides, the expressions for the integral of the square of the vortex field (enstrophy) and for all higher moments of the vortex field are obtained for the exact solution of the vortex 3D Euler equation (1.6). This corresponds to the establishment of an exact solution of the well-known closure problem in the theory of turbulence, to the solution of which only approximate approaches have been developed by W. Heisenberg, A.N. Kolmogorov et al. earlier.

THANK YOU FOR ATTENTION!